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Anathematizing the Guralnik-Manohar Bound for $\overline{\Lambda}$

I.I.Bigi

TH Division, CERN, CH-1211 Geneva 23, Switzerland [1]

and

Dept.of Physics, University of Notre Dame du Lac, Notre Dame, IN 46556, U.S.A. [2]

e-mail address: VXCERN::IBIGI, BIGI@UNDHEP

N.G.Uraltsev

TH Division, CERN, CH-1211 Geneva 23, Switzerland

and

St.Petersburg Nuclear Physics Institute, Gatchina, St.Petersburg 188350, Russia [2]

e-mail address: VXCERN::URALTSEV, URALTSEV@LNPI.SPB.SU

Abstract

There is a recent claim by Guralnik and Manohar [1] to have established a rigorous lower bound on $\bar{\Lambda}$, the asymptotic difference between the mass of a heavy flavour *hadron* and that of the heavy flavour *quark*. We point out the flaw in their reasoning and discuss the underlying physical problem. An explicit counterexample to the GM bound is given; one can therefore not count on a refined proof to re-establish this bound.

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[1]During the academic year 1993/94

[2]Permanent address

Very significant progress has been made in the last few years in the theoretical description of heavy flavour hadrons: in particular one has gained more control over the non-perturbative effects in static quantities as well as in some decay processes. The progress in the latter has come from an extensive application of Effective Heavy Quark Theory (EHQT) [2] to certain exclusive semileptonic modes on the one hand, and on the other from a direct QCD treatment [3-7] of inclusive decays through an expansion in powers of $1/m_Q$ with m_Q being the heavy flavour quark mass. In particular, the question of the leading non-perturbative corrections to inclusive transition rates has been addressed in these QCD papers: it was shown there that corrections of order $1/m_Q$ are absent in many instances, for example in total widths; they do however emerge in other cases, namely in the shape of semileptonic spectra or even in such integrated quantities like the average invariant mass of the hadronic final state for semileptonic or radiative decays [1].

There is one quantity of particular interest since it controls the leading non-perturbative effects for some decay rates and at the same time plays a fundamental role in EHQT: it is usually referred to as $\overline{\Lambda}$ and defined as the difference in the mass of a heavy flavour hadron H_Q and of the corresponding heavy quark Q in the limit $m_Q \to \infty$:

$$\langle H_Q|\bar{Q}\frac{1}{2}\,(i\nabla_\mu\gamma_\nu + i\nabla_\nu\gamma_\mu)Q|H_Q\rangle = (1 - \overline{\Lambda}/M_{H_Q}) \cdot \langle H_Q|T_{\mu\nu}|H_Q\rangle \tag{1}$$

where $T_{\mu\nu}$ denotes the full energy-momentum tensor in QCD whereas the operator on the *lhs* represents only the part associated with the heavy quark. The value of $\overline{\Lambda}$ is not known for sure even for the family of the lightest pseudoscalar mesons; reasonable estimates center around the value $\overline{\Lambda}_P \sim 300 \div 500\,\mathrm{MeV}$.

It has been claimed in a recent paper by Guralnik and Manohar (hereafter referred to as GM) that one can derive a rigorous lower bound $\overline{\Lambda} \geq 237\,\mathrm{MeV}$ for the heavy flavour pseudoscalar mesons. The aim of the present note is to show that this conclusion is erroneous: the actual proof given in ref.[1] is incorrect and its flaw in all likelyhood cannot be cured in any reasonable way. We will also present an explicit counterexample to the GM bound. Moreover we will give arguments that no sensible *rigouros* bound can be obtained at all for $\overline{\Lambda}$; at the same time some physically relevant statements can be made.

The idea underlying the derivation of ref. [1] is to use existing QCD mass inequality methods [9, 10] for hadrons containing heavy quarks. A straightforward application of such inequalities yields according to eq.(12) of GM the following bound:

$$M(\bar{Q}q) \geq \frac{1}{2}[m'(\bar{Q}i\gamma_5 Q) + m'(\bar{q}i\gamma_5 q)] \tag{2}$$

(the prime indicates that the annihilation contribution has been ignored, i.e. the masses stand for hypothetical mesons with non-identical, though mass-degenerate quarks and antiquarks); the inequality holds irrespective of the quantum numbers of the $\bar{Q}q$ state. This inequality *per se* does not lead to any bound on $\overline{\Lambda}$ due to the

[1] The problem of $1/m_Q$ effects was briefly addressed already in ref. [8] with respect to semileptonic decays only; however incorrect conclusions were stated about the existence of these corrections.

dominant, $\sim \alpha_s^2 \cdot m_Q$, negative Coulomb energy of the $\bar{Q}Q$ state on the *rhs*. Another inequality,

$$M(\bar{Q}q) - m_Q \geq 1/2\, m'(\bar{q}i\gamma_5 q) \tag{3}$$

was stated in eq.(11) of GM using arguments based on the effective theory for the infinitely heavy quark Q. This inequality differs from the previous one by the absence of the binding energy for the heavy quarkonium state and represents the main result of ref. [1]. We will show now that this bound is physically irrelevant.

The problem can be formulated in short as follows: in the *effective* theory of an *infinitely* heavy quark there exists no room for a quantity m_Q; therefore any results obtained in such a theory cannot be related to the difference between the masses of hadrons and the mass of the heavy quark. To be more specific: while the energy of a hadron state in the effective theory can be measured, say, in lattice simulations, it has in fact no direct connection to $\overline{\Lambda}$. Therefore the inequality stated in eq.(10) of GM does not lead to an inequality for $\overline{\Lambda}$ as claimed in eq.(11) of GM.

To clarify this at first sight paradoxical comment one needs to examine the subtleties of the derivation in more details. More specifically we shall now discuss the question of how the inequality for the correlators of the heavy quark currents

$$|\langle [\bar{q}\Gamma^+ Q](y)\, [\bar{Q}\Gamma q](x)\rangle|^2 \leq C \cdot \langle [\bar{q}i\gamma_5 q](x)\, [\bar{q}\gamma_5 q](y)\rangle \tag{4}$$

that was given in eq.(10) of GM can be used – and as it turns out, actually cannot be used – to arrive at eq.(3). Eq.(4), when considered for $\vec{x} = \vec{y}$ and a large Euclidean time difference $|x_4 - y_4| \to \infty$ states that the energy of $\bar{Q}q$ states in the effective theory is not less than half the energy of the lowest lying $\bar{q}q$ state. The question is how can this result be related to $\overline{\Lambda}$.

GM consider a lattice version of the theory originally containing the 'real' heavy quark Q with some finite mass m_Q. To obtain the Q propagator corresponding to the one in EHQT one then has to consider the limit

$$m_Q \gg 1/a \tag{5}$$

with a denoting the lattice spacing. Doing so one indeed arrives at eq.(3). However it is obvious that the mass m_Q entering this equation is the bare mass, m_Q^{bare}, that was originally present in the lattice theory. On the other hand the mass that enters the definition of $\overline{\Lambda}$ is not the bare one but rather the *pole* (or 'on-shell') mass.

In real QCD the two masses, namely the pole and the bare mass, differ by an infinite amount $\sim \alpha_s/\pi \cdot m_Q \log \Lambda_{\rm uv}^2/m_b^2$ where $\Lambda_{\rm uv}$ denotes some ultraviolet cutoff. In lattice QCD where there is a built-in maximal momentum scale of order $1/a$, there is a finite difference between the two masses. In the limit under consideration – $m_Q \gg 1/a$ – this difference is simply given by the classical Coulomb self-energy of a charged particle producing electrostatic fields:

$$m_Q^{\rm pole} - m_Q^{\rm bare} \simeq \frac{2}{3}\alpha_s/R \sim \alpha_s/a \tag{6}$$

where R is the 'radius' of the charged particle, $R \sim a$; it is important that the *rhs* of eq.(6) is necessarily positive. The bound for the quantity of interest, $\overline{\Lambda}$ then reads as

$$\overline{\Lambda}_{\bar{Q}q} \geq -c\, \alpha_s/a + 1/2\, m'(\bar{q}i\gamma_5 q) \tag{7}$$

with $c > 0$, i.e. with an additional negative term on the *rhs* relative to eq.(3).

How essential is this modification? The bound in eq.(7) of GM indeed is much stronger than the inequality in eq.(2) obtained in the standard way [9, 10]. However to be able to incorporate the light degrees of freedom of QCD one has to assume that $1/a \gg \Lambda_{\rm QCD}$. Then one realizes that the effective theory still gives an inequality that is trivially fulfilled: although the *rhs* of the equation does not scale like m_Q, it still does not provide a useful lower bound: for the first term is parametrically larger than $\Lambda_{\rm QCD}$ and negative!

At first sight it would seem one can try to evade this problem by a natural modification in the line of reasoning, namely by considering a smaller lattice spacing in order to reach the continuum limit even for the heavy quarks. Unfortunately that does not work either, as revealed by closer scrutiny. For as soon as the heavy quarks become non-stationary through interactions with the gluon fields that are non-leading in $1/m_Q$, the average of the product of the two heavy quark propagators W in in eq.(7) of GM is no longer bounded from above by a constant – instead it grows exponentially with time. These corrections are generated by extra terms in the heavy quark propagators that appear in eq.(3) of GM. This positive exponent is a reflection of the (attractive) interaction between the propagating heavy quarks that is mediated by the gluon field. Then additional negative terms appear in the *rhs* of the inequality for $\overline{\Lambda}$ and thus the GM inequality gets invalidated. The origin of the additional interaction for propagating heavy quarks is very lucidly discussed in GM and we do not need to address it here.

It was alluded in GM to a different treatment of the effective theory that leads to the same inequality. Since that proof has not been given in GM we cannot of course point out at which point exactly it goes awry. Instead we will present a concrete counterexample to the basic inequality. This will demonstrate that this bound cannot be resurrected by a more clever proof.

Let us consider muonium in QED, i.e. the boundstate of a muon and a positron in a hydrogen-like system. This represents a 'Heavy Fermion' scenario with the added bonus that the boundstate properties can be calculated explicitly; the muon mass provides the high mass scale and m_e, the positron mass, sets the scale for the 'light degrees of freedom' in the EHQT [2]. All the general arguments employed in GM apply here directly and therefore one would obtain the inequality of eq.(3).

The mass spectrum in such a theory is well known: the boundstate mass equals the sum of the *pole* masses of the constituents plus the Coulomb binding energy $E_C = -\alpha^2/2 \cdot m_{red}$ where m_{red} denotes the reduced mass. Both sides of the inequality can thus be evaluated leading to the claim

$$m_\mu + m_e - \frac{\alpha^2}{2} m_e - m_\mu \geq \frac{1}{2} \left(2m_e - \frac{\alpha^2}{2} \frac{m_e}{2} \right) \tag{8}$$

which is obviously incorrect.

This explicit counterexample demonstrates that the bound claimed in GM cannot be established rigorously, independently of possible technical modifications of the concrete proof. We believe that this 'no-go' statement is not accidental, but reflects

[2] Likewise one can also consider real QCD with two heavy quarks, namely b and c with $m_b \gg m_c \gg \Lambda_{QCD}$.

some deeper reasons. Although our arguments are rather general and cannot be considered as any kind of rigorous statement, we shall discuss them now.

The main difficulty in making precise statements about $\overline{\Lambda}$ in terms of, say, $\Lambda_{\rm QCD}$ is that $\overline{\Lambda}$ is not well defined in terms of the *bare* parameters of QCD. While the masses of hadrons constitute clearly unambiguous quantities, the renormalized mass of the heavy quark does not; for it suffers from perturbative corrections having a *multiplicative* form:

$$m_Q^{\rm pole} = m_Q + c_1 \alpha_s(m_Q^2)\, m_Q + c_2 \alpha_s^2(m_Q^2)\, m_Q + \ldots \quad ; \tag{9}$$

the generic term in the series has then the form $m_Q/\log^n(m_Q/\Lambda_{\rm QCD})$. This series is asymptotic, and any term in it must be considered to be parametrically larger than the quantity of interest, $\overline{\Lambda}$, in the limit $m_Q \to \infty$. From a formal point of view the result for $\overline{\Lambda}$ thus depends completely on the way how the infinite series is summed up.

The arguments given above may seem to suggest that any constructive definition of $m_Q^{\rm pole}$ and therefore of $\overline{\Lambda}$ is impossible with an accuracy [3] of better than $\sim m_Q$. The conventional argument would be that due to colour confinement the asymptotic states corresponding to free quarks do not exist; therefore whatever accurate method of treating QCD is developed in the future, the pole mass cannot be defined. We believe that such agnosticism overstates the real problem. For one can consider that phase of QCD where the gauge symmetry is spontaneously broken by some Higgs fields at the relatively low scale $\gtrsim \Lambda_{\rm QCD}$ which is much smaller than m_Q. The energy of the deconfined single heavy quark states do depend of course on the details of the Higgs sector in such theories; however the scale for the variation in the mass is given by $\Lambda_{\rm QCD}$ and not by m_Q. The pole mass of the heavy quark defined in such a way must identically coincide with the standard definition in QCD to all orders of the perturbative expansion.

One sees therefore that a constructive definition of the $m_Q^{\rm pole}$ can be given in a straightforward way even beyond perturbation theory. However it neccessarily suffers from an uncertainty of order $\Lambda_{\rm QCD}$; that is just the relevant scale for a discussion of $\overline{\Lambda}$. One of the possible such definitions is of course to state that $m_Q^{\rm pole}$ is simply the mass of, say, the lightest pseudoscalar $\bar{Q}q$ state with a massless spectator. This definition satisfies all necessary requirements and is of course completely constructive, however the question about the size of $\overline{\Lambda}$ becomes a tautology.

One could think of somehow defining the heavy quark pole mass just via the lattice realization of QCD with heavy quarks as was implied in GM. Then the bound claimed in that paper would be rigorous for the pole mass defined in some specific way. Our analysis of the GM proof given above shows, however, that this definition can *not* satisfy the necessary requirements: it yields a pole mass which is *parametrically smaller* – by an amount $\sim 1/a \log(a)$ – than any of the 'correct' pole masses. Therefore any finite inequality derived for a thus defined mass is without physical relevance.

We thus arrive at the conclusion that from a formal point of view a description of powerlike nonperturbative effects makes sense only [3, 7, 11, 12, 13] for quantities that

[3] From now on we neglect factors of $\log(m_Q)$.

have no purely perturbative corrections in any finite order. These are for example mass differences among various heavy flavoured hadrons, inclusive width differences between mesons and baryons or width splittings among the members of multiplets, differences in spectra etc. These effects are determined by the differences in hadronic quantities like $\overline{\Lambda}$ for different hadrons; after all they are expressed in terms of the masses (or other characteristics) of *hadrons* and are therefore unambiguous. One note of caution should be added here: in a quantum description it often happens that inequalities lose their validity through a neccessary subtraction procedure. For example, the effective values of such quantities as $\bar{Q}(i\vec{\mathbf{D}})^2Q$ – though naively positively defined – could in principle appear to be of either sign.

In practice of course one deals with the real world of hadrons where the masses of c and b quarks are rather large, but finite. It is just in such a situation when the analysis of the non-perturbative corrections which scale like $(1/m_Q)^n$ has practical relevance. It was shown that in many cases the leading non-perturbative corrections are at least as large numerically as the perturbative ones even for the beauty system [3, 7, 12] and it is a practical necessity to properly account for them. In particular the problem of the numerical value of m_b and m_c, or in other words, $\overline{\Lambda}$ is important for various phenomenological applications. On the other hand in practical calculations one accounts only for a few terms in both perturbative and nonperturbative expansions. This truncated procedure is justified for the case of a finite mass m_Q: nonperturbative corrections are to account for the physics of the low scale of order $\Lambda_{\rm QCD}$ whereas the perturbative ones reflect the dynamics at high momenta $\sim m_Q$. The regime of the perturbative corrections in a particular order may however cover in reality also momenta that constitute lower and lower fractions of m_Q; for a fixed m_Q at some order the two regions start to overlap thus violating the physical basis for the two expansions. This suggests the conclusion that for the real masses of b and c quarks non-perturbative quantities including $\overline{\Lambda}$ have a reasonable physical meaning, albeit with an intrinsic limitation on the possible accuracy of their values, both from a theoretical and phenomenological point of view (see also ref. [13] for a similar discussion). For example one may hope to get a reasonably accurate *numerical* estimate of $\overline{\Lambda}$ from QCD sum rules (see e.g. refs. [14, 15]) or lattice calculations at small, but finite lattice spacing. Needless to say that such a quantity may depend significantly on the exact way it is defined and to which order in the perturbative expansion one has gone.

To summarize: we have shown that the derivation of the bound on $\overline{\Lambda}$ obtained in ref.[1] is fundamentally flawed if applied to the real physical quantity. It actually may refer only to some different quantity rather than $\overline{\Lambda}$ for which this inequality is satisfied trivially. We have argued that in all likelihood no rigorous bound of a similar type can be obtained in QCD. However physically justified statements about $\overline{\Lambda}$ can in principle be obtained by applying various dynamical consideration, with certain intrinsic limitations on the possible numerical precision.

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